

SIMPLIFIED ECE 174 Computer Assignment 2

GLOBAL POSITIONING SYSTEM (GPS) ALGORITHM

Overview

By utilizing measurements of the so-called pseudorange between an object and each of several earth orbiting GPS satellites, the object can be very accurately located in space. A general description of GPS can be found in the article [1], which you should read thoroughly, and in the detailed Web site [2]. Additional technical information can be found in [3] and other articles in [4]. Using your knowledge of the nonlinear least-squares problem you will solve the problem of locating an object given GPS pseudorange data. A self-contained technical development, sufficient for you to perform this task, can be found in the appendix to this assignment and in the class lecture supplement on gradient descent GPS algorithms [5].

Signals are sent from earth orbiting satellites to a receiver located on the surface of the earth. In principle, utilizing timing information and knowledge of the satellite orbit, the true range (the receiver-to-satellite distance) to each satellite can be measured. In practice the range measurement is errorful due to a systematic error caused by an inaccurate receiver clock (to keep the cost of commercial GPS units low) and random errors due to atmospheric distortion, communication channel noise, sensor noise, etc. The errorful range measurement is known as a *pseudorange* in order to distinguish it from the unknown true range (see the Appendix below). Because of the speed of light; the non-relativistic speeds of the satellites and receiver; and the close proximity of the satellites and object, it is assumed that during the time period needed to collect all the pseudorange information the satellites and receiver are motionless.

Notation and Problem Statement

All locations are given with respect to a fixed-earth (geostationary) reference coordinate system located at the center of the earth. Distances are given in units of *Earth Radii* (ER) (using an average value of $1ER = 6,370km$). Note that according to [1], GPS satellites are located at about $4.14ER$ from the *center* of the earth ($20,000km$ above the earth's *surface*).

Use the notation $S_\ell = (s_{\ell,1}, s_{\ell,2}, s_{\ell,3})^T$ = position of ℓ^{th} Satellite, $\ell = 1, \dots, 4$; $S = (s_1, s_2, s_3)^T$ = position of receiver (the Station to be tracked); R_ℓ = true range to the receiver located at S from the satellite located at S_ℓ ; and $\Delta S_\ell = S - S_\ell$ = satellite-to-receiver distance vector. For fixed satellite location, S_ℓ , the true range, R_ℓ , is a nonlinear function of the receiver location, S , and is given by,

$$R_\ell(S) = \|S - S_\ell\| = \|\Delta S_\ell\| = \sqrt{(\Delta S_\ell)^T \Delta S_\ell}, \quad \ell = 1, \dots, 4.$$

The pseudorange measurements are denoted by y_ℓ , and modeled as¹

$$y_\ell = R_\ell(S) + b + \nu_\ell, \quad \ell = 1, \dots, 4, \tag{1}$$

¹See the attached appendix.

where the random noise term ν_ℓ is i.i.d. with p.d.f. $N(0, \sigma^2)$. Further discussion of this model is given in the lecture supplement [5]. The (constant) systematic *clock bias* error b is caused by an inaccurate clock in the GPS receiver. The number of range measurements taken to each satellite, m , is the same for each satellite. We will take the simple case of one range measurement for each satellite, $m = 1$.

- Knowing the satellite locations, S_ℓ , $\ell = 1, \dots, 4$, and having m pseudorange measurements, y_ℓ , to each of the ℓ satellites, you are to estimate the receiver location, S , and the clock bias, b , using the Gradient Descent and Gauss methods for solving nonlinear least squares problems.

Simulation Parameter Values. For the purposes of generating synthetic data, the actual receiver position, $S = S_0$, and the satellite positions, S_ℓ , are taken to be as follows (in ER units):

$$\begin{aligned} S &= (1.000000000, 0.000000000, 0.000000000)^T \\ S_1 &= (3.585200000, 2.070000000, 0.000000000)^T \\ S_2 &= (2.927400000, 2.927400000, 0.000000000)^T \\ S_3 &= (2.661200000, 0.000000000, 3.171200000)^T \\ S_4 &= (1.415900000, 0.000000000, 3.890400000)^T. \end{aligned}$$

The receiver is assumed to be at mean sea level. The clock bias error, b , is taken to be

$$b = 2.354788068 \times 10^{-3} ER,$$

which is equivalent to $b = 15 km$, or approximately 0.05 millisecond.² As discussed in the appendix, $b > 0$ corresponds to the clock on the receiver running fast compared to the clocks on the satellites. Note the degree of numerical round-off precision required to locate an object to well within a kilometer ($1km = 1.569858713 \times 10^{-4} ER$). By default Matlab uses double precision to minimize numerical round-off error, which should be adequate for our problem.

Procedure

1. **Linearization.** Assume a nominal receiver location, \hat{S} , and clock bias \hat{b} and linearize the pseudorange equation (1) about the nominal values. Discuss whether the linearized approximation gets better or worse as the true range increases.
2. **Algorithm Development.** Formulate the *Steepest Descent* and *Gauss'* methods, explicitly showing the actual equations you will use. Identify the step size you will use. Identify your termination criteria. Remember that, in general, it is assumed that m pseudorange measurements have been taken to each satellite. To accomplish this step, you will need to carefully read and understand the material presented in Lecture Supplements 2 and 3.

²Using $c = 299,792.458$ km/sec. The specified value of b , then, is the instantaneous clock accuracy needed to locate an object to within 15 kilometers. Note that to attain the accuracy needed to locate an object to within 10 meters requires that we reduce this clock imprecision by a factor of at least $1/1,500$.

3. **Simulation.** Generate synthetic data and test and compare the Steepest Descent and Gauss' Methods. For your initial estimate of the vehicle location, $\hat{S}(0)$, use

$$\hat{S}(0) = (0.93310, 0.25000, 0.258819)^T ER.$$

For your initial estimate of the clock bias use $\hat{b}(0) = 0$. The initial estimate $\hat{S}(0)$ you are to use corresponds to a location which is about 0.5km above sea level and about 2,330km off of the actual location S . Note that this is a *very* crude initial guess—as a comparison, the distance between San Diego and Los Angeles is less than 200km.

Now generate synthetic data, only for the zero noise ($\nu_\ell = 0 \iff \sigma = 0$) case:

$$y_\ell := R_\ell(S_0) + b, \quad \ell = 1, 2, 3, 4.$$

Now use the generated synthetic data and the mathematical model

$$y_\ell = R_\ell(S) + b, \quad \ell = 1, 2, 3, 4,$$

with S a free vector variable. to obtain an estimate of the true (assumed unknown) position S_0 as follows:

- (a) Apply the *Steepest Descent Algorithm* to locate the vehicle for the zero noise case. Show the errors in your position estimates using units of *meters*.
 - (b) Repeat step (a) for the *Gauss-Newton Algorithm*. Again, use units of *meters* in the error analysis.
4. **Discussion.** Write up your results using our standard format. Show and discuss the results of Steps 1–3 in some detail, including the observed convergence rates of steepest descent versus the Gauss-Newton method. Present plots showing the loss function $\ell(k)$, receiver position estimate error $\|\hat{S}(k) - S\|$ in units of **meters**, and clock bias estimate error $|\hat{b}(k) - b|$ also in units of **meters**, as a function of iteration step, k .

By ignoring noise in the generation of your synthetic data (the noise represents modeling errors, which **always** occur in the real world), you will get an *artificially good* GPS estimate of your location. For this reason the simplified version of this assignment produces *unreasonably* optimistic results.

References

- [1] “The Global Positioning System,” Thomas A. Herring, *Scientific American*, Vol. 274, No. 2, February 1996, pp. 44-50. Available from UCSD S&E Library Electronic Archives.
- [2] “GPS: Global Positioning System Overview,” Peter H. Dana, Department of Geology, UT Austin, located at <http://www.colorado.edu/geography/gcraft/notes/gps/gps.html>
- [3] “GPS Performance in Navigation,” P. Misra, B.P. Burke, and M.M. Pratt, in reference [4], pp. 65–85.

- [4] “Special Issue on GPS: The Global Positioning System,” *Proceedings of the IEEE*, January 1999, Vol. 87, No. 1. Available from IEEE Xplore at <http://ieeexplore.ieee.org/Xplore/dynhome.jsp>
- [5] “Gradient–Descent GPS Algorithms,” ECE174 Class Lecture Supplement, K. Kreutz-Delgado, ECE Department, UCSD.

APPENDIX: The Pseudorange Equation.

Let us focus on a single satellite (say the ℓ -th one). At true time t_ℓ the satellite transmits to the receiver its precise location in space, S_ℓ , and the time, t_ℓ , of transmission. The clocks on the satellites are very expensive atomic clocks of the highest accuracy. Because of the quality of the atomic clocks on the satellites, we assume that they all agree and are error free.

Because of the finite (albeit extremely high) speed of light, the signal from satellite ℓ is received at a slightly later time $t > t_\ell$, where t is the true time of reception. The true elapsed time of propagation of the signal is therefore $t - t_\ell$. Denoting the speed of light **in a vacuum** by c , in principle we can determine the range, R_ℓ , between the satellite and the receiver from

$$\text{range} = c(t - t_\ell) \quad (\text{ideally}) \quad (2)$$

assuming that c is constant along the communication path and that the path is a straight line. This equation is exact in the vacuum of space. However, in fact atmospheric propagation effects affect this assumption. In reality, **there is an effective (average) speed of light** between the satellite location S_ℓ and the receiver location S , $c_{\ell,\text{eff}}$ such that the correct equation is

$$R_\ell = R_\ell(S) = c_{\ell,\text{eff}}(t - t_\ell) \quad (\text{True Range}) \quad (3)$$

However, we do not know $c_{\ell,\text{eff}}$ (which changes with the atmospheric conditions) so we can only make an estimate of the range via using the value of the speed of light in a vacuum c . If we define the speed-of-light error $\delta c_\ell = c - c_{\ell,\text{eff}}$ then we have

$$c = c_{\ell,\text{eff}} + \delta c_\ell. \quad (4)$$

Things are even worse than this because we also don't true the exact value of the true time of reception t . This is because (unfortunately from a modeling perspective but fortunately for the consumer!) the clock located at the receiver (e.g., in your smart phone) is not an expensive atomic clock, but rather an affordable clock chosen to keep the cost of the receiver down. Thus, in general the receiver believes that it receives the signal at an erroneous time t' ($t' \neq t$) according to the cheap clock located at the receiver. The receiver clock synchronization error can be defined as $\Delta t = t' - t$. Thus

$$t' = t + \Delta t. \quad (5)$$

If $\Delta t = t' - t > 0$, the receiver clock is “fast” (i.e., it “ticks” too fast), while if $\Delta t = t' - t < 0$, the receiver clock is “slow.” The “pseudo-elapsed time” is given by $t' - t_\ell > 0$ (which we

assume is positive) and corresponds to the time of transit of the signal as determined by the cheap receiver clock. Thus the receiver calculates the range (erroneously) as

$$y_\ell = c(t' - t_\ell). \quad (6)$$

Based on our discussion, we know that both c and t' are the incorrect values to use, so y_ℓ provides an incorrect approximation to the true range R_ℓ . The approximation y_ℓ is called the **pseudorange**. We emphasize: The error in the pseudorange y_ℓ is due to two causes: the erroneous use of the speed of light in a vacuum, c , and the erroneous use of the receiver clock time t' .

Let us proceed find a mathematical relationship between the pseudorange y_ℓ and the true range R_ℓ . First we note that

$$y_\ell = c(t' - t_\ell) = c(t - t_\ell + \Delta t) = c(t - t_\ell) + b, \quad (7)$$

where

$$b \triangleq c\Delta t = c(t' - t). \quad (8)$$

The real number b is *defined to be* the clock bias in units of distance that light travels in a vacuum during the clock bias time interval $\Delta t = t' - t$. Note that b has units of length and that, as defined, it is independent of satellite ℓ .³ Also note that once a good estimate of b is available, \hat{b} , we can determine an estimate of the clock synchronization error in units of time as $\hat{\Delta t} = c^{-1}\hat{b}$ so that we can obtain an estimate of the exact time $t \approx t' - \hat{\Delta t}$ that almost as good as its actual value $t = t' - \Delta t$. I.e., the GPS algorithm can be used to make the cheap clock in the receiver effectively into a much more accurate clock.

Continuing from Eq. (5), we have

$$y_\ell = (c_{\ell,\text{eff}} + \delta c_\ell)(t - t_\ell) + b = \underbrace{c_{\ell,\text{eff}}(t - t_\ell)}_{R_\ell} + b + \underbrace{(\delta c_\ell)(t - t_\ell)}_{\nu_\ell},$$

where we assume that $\nu_\ell \sim \mathcal{N}(0, \sigma^2)$. Thus we have arrived at the pseudorange measurement model relating the pseudorange y_ℓ to the true range R_ℓ ,

$$y_\ell = c(t' - t_\ell) = R_\ell(S) + b + \nu_\ell, \quad (9)$$

which was earlier presented in (1). The information packet sent from satellite ℓ at time t_ℓ (containing the data t_ℓ and S_ℓ) and clocked as being received by the receiver at time t' enables us to compute the pseudorange via $y_\ell = c(t' - t_\ell)$, and to construct the functions

$$\frac{\partial R_\ell(S)}{\partial S} = \frac{\partial}{\partial S} \|S - S_\ell\| = \frac{\partial}{\partial S} \sqrt{(S - S_\ell)^T (S - S_\ell)}$$

needed to perform the least squares estimation.

³As it should be. Whether or not the clock in your cell phone runs fast or slow is independent of what a satellite thousands of miles away happens to be doing. Note that *no* approximation has been made in Eq. (7).